Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

On the Chermak–Delgado lattice of a finite group

Ryan McCulloch & Marius Tărnăuceanu

COMMUNICATIONS IN ALGEBRA®
2020, VOL. 48, NO. 1, 37–44
https://doi.org/10.1080/00927872.2019.1632326

Taylor & Francis
Taylor & Francis Group

Check for updates

On the Chermak–Delgado lattice of a finite group

Ryan McCulloch[a] and Marius Tărnăuceanu[b]

[a]Department of Mathematics, University of Bridgeport, Bridgeport, CT, USA; [b]Faculty of Mathematics, "Al.I. Cuza" University, Iaşi, Romania"

ABSTRACT
By imposing conditions upon the index of a self-centralizing subgroup of a group, and upon the index of the center of the group, we are able to classify the Chermak-Delgado lattice of the group. This is our main result. We use this result to classify the Chermak-Delgado lattices of dicyclic groups and of metabelian p-groups of maximal class.

ARTICLE HISTORY
Received 14 September 2018
Revised 29 May 2019
Communicated by
Pham Huu Tiep

KEYWORDS
Chermak–Delgado measure; Chermak–Delgado lattice; Chermak–Delgado subgroup; subgroup lattice; (generalized) dicyclic group; metabelian group

MSC 2000
Primary 20D30; Secondary 20D60; 20D99

1. Introduction

Throughout this article, G will denote a finite group. Denote by

$$m_G(H) = |H||C_G(H)|$$

the *Chermak–Delgado measure* of a subgroup H of G and let

$$m^*(G) = \max\{m_G(H)|H \leq G\} \text{ and } \mathcal{CD}(G) = \{H \leq G \,|\, m_G(H) = m^*(G)\}.$$

The set $\mathcal{CD}(G)$ forms a modular, self-dual sublattice of the lattice of subgroups of G, which is called the *Chermak–Delgado lattice* of G. It was first introduced by Chermak and Delgado [5], and revisited by Isaacs [7]. In the last years there has been a growing interest in understanding this lattice (see e.g. [1–4, 6, 8–10, 13–15, 18]). Recall two important properties of the Chermak–Delgado lattice that will be used in our article:

- if $H \in \mathcal{CD}(G)$, then $C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$;
- the minimum subgroup $M(G)$ of $\mathcal{CD}(G)$ (called the *Chermak–Delgado subgroup* of G) is characteristic, abelian, and contains $Z(G)$.

For a positive integer $n \geq 1$, the dicyclic group of order $4n$, usually denoted by Dic_{4n}, is defined as

$$Dic_{4n} = \langle a, x \mid a^{2n} = 1, x^2 = a^n, a^x = a^{-1}\rangle.$$

CONTACT Marius Tărnăuceanu tarnauc@uaic.ro Faculty of Mathematics, "Al.I. Cuza" University, Iaşi, Romania.

This has the following generalization: given an arbitrary abelian group A of order $2n$, the generalized dicyclic group induced by A is defined as

$$Dic_{4n}(A) = \langle A, x \mid x^4 = 1, x^2 \in A \setminus \{1\}, a^x = a^{-1}, \forall\ a \in A \rangle.$$

Obviously, we have $Dic_{4n}(\mathbb{Z}_{2n}) = Dic_{4n}$. It is also easy to see that if $\exp(A) = 2$, that is that A is an elementary abelian 2-group, then $Dic_{4n}(A)$ is abelian and consequently $\mathcal{CD}(Dic_{4n}(A)) = \{Dic_{4n}(A)\}$. Note that if $\exp(A) \neq 2$, then

$$Z(Dic_{4n}(A)) = \{a \in A \mid a^2 = 1\} \cong \frac{A}{A^2} \ ,$$

where $A^2 = \{a^2 \mid a \in A\}$.

A finite group G is said to be *metabelian* if the derived subgroup, G', is abelian. Equivalently, a finite group G is metabelian if there exists an abelian normal subgroup A of G so that G/A is abelian.

A finite p-group G of order p^n is said to be *of maximal class* if the nilpotence class of G is $n - 1$. The following results on p-groups will be useful to us. Lemma 1.1 appears in (4.26), [12], II, Lemma 1.2 appears in Theorem 2.4 of [16] (see also (12), [11], p. 392), and Lemma 1.3 at end of [14].

Lemma 1.1. *Any group of order p^4 contains an abelian subgroup of order p^3.*

Lemma 1.2. *Suppose G is a p-group of order p^n and G is of maximal class. Then $|Z(G)| = p, |G : G'| = p^2$, and for each $2 \leq i \leq n$, we have that G_i is the unique normal subgroup of G order p^{n-i}, where G_i is the ith term in the lower central series for G.*

Lemma 1.3. *Let G be a p-group of maximal class and of order p^5. If $m^*(G) = p^6$, then $\mathcal{CD}(G) = \{G, T, G', A_1, ..., A_p, Z(T), Z(G)\}$, where $|T| = p^4, |G'| = |A_i| = p^3$ for each i, $|Z(T)| = p^2$, and $G', A_1, ..., A_p$ are all abelian and distinct. Also, none of $A_1, ..., A_p$ are normal in G.*

2. Main results

In this section we present a result which generalizes Proposition 7 in [10]. It will be used both in Sections 3 and 4. A subgroup A of G is said to be self-centralizing if $C_G(A) = A$. An important observation used here is that if $T \in \mathcal{CD}(G)$, then since $Z(G) \leq T$, we have that $|G : T|$ divides $|G : Z(G)|$. So by imposing conditions on $|G : Z(G)|$, we are able to obtain results about $\mathcal{CD}(G)$.

Theorem 2.1. *Let G be a finite group, let p be a prime, and among the self-centralizing subgroups of G, let A be one of maximum order.*

1. *If G is abelian, then $\mathcal{CD}(G) = \{G\}$.*
2. *If $|G : Z(G)| = p^2$, then $\mathcal{CD}(G) = \{Z(G), A, A_1, ..., A_p, G\}$ is a quasi-antichain of width $p + 1$, with each A_i abelian.*
3. *If $|G : A| = p$ and $|G : Z(G)| = p^i$ with $i > 2$, then $\mathcal{CD}(G) = \{A\}$.*
   If p is the smallest prime divisor of $|G|$ and $|G : A| = p$ and $|G : Z(G)| > p^2$, then $\mathcal{CD}(G) = \{A\}$.
4. *Suppose $|G : A| = p^2$.*
   a. *If $|G : Z(G)| = p^3$, then $\mathcal{CD}(G) = \{Z(G), G\}$.*
   If p is the smallest prime *divisor of $|G|$ and $p^2 < |G : Z(G)| < p^4$, then $\mathcal{CD}(G) = \{Z(G), G\}$.*
   b. *If $|G : Z(G)| = p^4$, then*

   $$\mathcal{CD}(G) = \{Z(G), Z(T_1), ..., Z(T_n), A, A_1, ..., A_m, T_1, ..., T_n, G\}$$

   where $T_1, ..., T_n$ $(n \geq 0)$ are all of the subgroups of index p in G with centers that have index p^3 in G, and $A_1, ..., A_m$ $(m \geq 0)$ are all of the subgroups (other than A) of index p^2 in G with centralizers that have index p^2 in G. Furthermore, if $n \geq 1$, then $m \geq p$.

c. *If $|G:Z(G)| = p^i$ with $i > 4$ and if G possesses a subgroup, T, of index p in G with center that has index p^3 in G, then $\mathcal{CD}(G) = \{Z(T), A, A_1, ..., A_p, T\}$ is a quasi-antichain of width $p+1$, with each A_i abelian.*
If p is the smallest prime divisor of $|G|$ and if $|G:Z(G)| > p^4$ and if G possesses a subgroup, T, of index p in G with center that has index p^3 in G, then $\mathcal{CD}(G) = \{Z(T), A, A_1, ..., A_p, T\}$ is a quasi-antichain of width $p+1$, with each A_i abelian.

d. *If $|G:Z(G)| = p^i$ with $i > 4$ and if G does not possess a subgroup, T, of index p in G with center that has index p^3 in G, then $\mathcal{CD}(G) = \{A\}$.*
If p is the smallest prime divisor of $|G|$ and if $|G:Z(G)| > p^4$ and if G does not possess a subgroup, T, of index p in G with center that has index p^3 in G, then $\mathcal{CD}(G) = \{A\}$.

Proof. Item 1 is clear.

To see item 2, note first that $|G:G||G:Z(G)| = |G:Z(G)| = p^2 \leq |G:T||G:C_G(T)|$ for any subgroup T of G with $Z(G) \leq T$, and so $G, Z(G) \in \mathcal{CD}(G)$. Note that $G/Z(G) \cong C_p \times C_p$, and so there are exactly $p+1$ subgroups H with $Z(G) < H < G$. For any such H, we have that $H = \langle Z(G), x\rangle$ for some $x \in H \setminus Z(G)$, and thus H is abelian. It follows that $\mathcal{CD}(G) = \{Z(G), A, A_1, ..., A_p, G\}$ is a quasi-antichain of width $p+1$, with each A_i abelian.

We prove both parts of item 3 simultaneously. Note that $|G:G||G:Z(G)| = |G:Z(G)| > p^2 = |G:A||G:A|$, and so $G, Z(G) \notin \mathcal{CD}(G)$. Note that $|G:A||G:A| = p^2 \leq |G:T||G:C_G(T)|$ for any subgroup T of G with $Z(G) \leq T$, and so $A \in \mathcal{CD}(G)$. Now A is a maximal subgroup of G and $G \notin \mathcal{CD}(G)$, and so A is the largest member of $\mathcal{CD}(G)$. But A is self-centralizing and therefore A is also the least member of $\mathcal{CD}(G)$. Hence $\mathcal{CD}(G) = \{A\}$.

We prove both parts of item 4.a simultaneously. Note that $|G:A||G:A| = p^4 > |G:Z(G)| = |G:G||G:Z(G)|$, and so $A \notin \mathcal{CD}(G)$. Since among the self-centralizing subgroups of G, A is one of maximum order, it follows that $\mathcal{CD}(G)$ does not contain any self-centralizing subgroups. Among the abelian subgroups of $\mathcal{CD}(G)$, let T be a maximal one. If $T \neq Z(G)$, then we have that $T < C_G(T) < G$, and furthermore we have that $|C_G(T):T|$ cannot be a prime. This is true because if $|C_G(T):T|$ is a prime, then $C_G(T)/T$ is cyclic, and hence $C_G(T)$ is abelian, and so T would not be maximal among the abelian subgroups in $\mathcal{CD}(G)$. It follows that $|G:T||G:C_G(T)| \geq p^4 > |G:G||G:Z(G)|$, a contradiction. Hence $T = Z(G)$ is the only abelian subgroup in $\mathcal{CD}(G)$. If $H \in \mathcal{CD}(G)$ is a subgroup of index p in G with $C_G(H)$ a subgroup of index p^2 in G, then $|C_G(H):Z(G)| < p^2$, and hence is prime. But then $C_G(H)$ is abelian, a contradiction. It follows that $\mathcal{CD}(G) = \{Z(G), G\}$.

To see item 4.b, note that $p^4 = |G:G||G:Z(G)| = |G:A||G:A|$. Since among the self-centralizing subgroups of G, A is one of maximum order, we have that any other self-centralizing subgroup A' of $\mathcal{CD}(G)$ will have that $p^4 = |G:A'||G:A'|$. If T is an abelian subgroup of $\mathcal{CD}(G)$ with $T < C_G(T) < G$ and $C_G(T)$ nonabelian, then as we saw in the proof of item 4.a, we have that $|G:T||G:C_G(T)| \geq p^4$. It follows then that $G, Z(G), A \in \mathcal{CD}(G)$. And hence

$$\{Z(G), Z(T_1), ..., Z(T_n), A, A_1, ..., A_m, T_1, ..., T_n, G\} \subseteq \mathcal{CD}(G)$$

where $T_1, ..., T_n$ ($n \geq 0$) are all of the subgroups of index p in G with centers that have index p^3 in G, and $A_1, ..., A_m$ ($m \geq 0$) are all of the subgroups (other than A) of index p^2 in G with centralizers that have index p^2 in G.

We also have that

$$\mathcal{CD}(G) \subseteq \{Z(G), Z(T_1), ..., Z(T_n), A, A_1, ..., A_m, T_1, ..., T_n, G\}.$$

This is true because, otherwise, if some $H \in \mathcal{CD}(G)$ has that $|G:H| = p$ and $|G:C_G(H)| = p^3$, but $Z(H) < C_G(H)$, then since $Z(H) = H \cap C_G(H) \in \mathcal{CD}(G)$, it would follow that $Z(H) = Z(G)$ and thus $H = G$, a contradiction.

If $n \geq 1$, then for some $T \in \{T_1, ..., T_n\}$, we have that $Z(T) < A < T$ in $\mathcal{CD}(G)$. This is true because, otherwise, we would have $Z(G) < A < G$ and $Z(G) < Z(T) < T < G$ as maximal chains in

$\mathcal{CD}(G)$, contradicting the modularity of $\mathcal{CD}(G)$. We can apply item 2 in Theorem 2.1 to the group T, to conclude that there are p additional self-centralizing groups in $\mathcal{CD}(T)$, and since all of these are self-centralizing, they are also all in $\mathcal{CD}(G)$.

We prove both parts of item 4.c simultaneously. Note that $|G:G||G:Z(G)|>p^4=|G:A||G:A|$, and so $G, Z(G) \notin \mathcal{CD}(G)$. As we saw before, if S is an abelian subgroup of $\mathcal{CD}(G)$ with $S<C_G(S)<G$ and $C_G(S)$ nonabelian, then $|G:S||G:C_G(S)|\geq p^4$. It follows that $A, T, Z(T) \in \mathcal{CD}(G)$, and since T is maximal in G, we have that T is the largest member of $\mathcal{CD}(G)$. And so applying Theorem 2.1 item 2 to the group T, we obtain that $\mathcal{CD}(G)=\{Z(T), A, A_1, \ldots, A_p, T\}$ is a quasi-antichain of width $p+1$, with each A_i abelian.

We prove both parts of item 4.d simultaneously. As we saw in the proof of item 4.c, $G, Z(G) \notin \mathcal{CD}(G)$ and $A \in \mathcal{CD}(G)$. But since there is no subgroup of index p in $\mathcal{CD}(G)$, we have that A is the largest member of $\mathcal{CD}(G)$, and since A is self-centralizing, $A=C_G(A)$ is the least member of $\mathcal{CD}(G)$, and we conclude that $\mathcal{CD}(G)=\{A\}$. □

Items 3, 4.c, and 4.d in Theorem 2.1 have interesting consequences due to the uniqueness of the largest member of the Chermak–Delgado lattice. We collect them below as a corollary.

Corollary 2.2. *Let G be a finite group, let p be a prime, and among the self-centralizing subgroups of G, let A be one of maximum order.*

1. *If $|G:A|=p$ and $|G:Z(G)|=p^i$ with $i>2$, then A is unique.*
   If p is the smallest prime divisor of $|G|$ and $|G:A|=p$ and $|G:Z(G)|>p^2$, then A is unique.
2. *If $|G:A|=p^2$ and $|G:Z(G)|=p^i$ with $i>4$ and if G possesses a subgroup, T, of index p in G with center that has index p^3 in G, then T is unique.*
   If p is the smallest prime divisor of $|G|$ and if $|G:A|=p^2$ and $|G:Z(G)|>p^4$ and if G possesses a subgroup, T, of index p in G with center that has index p^3 in G, then T is unique.
3. *If $|G:A|=p^2$ and $|G:Z(G)|=p^i$ with $i>4$ and if G does not possess a subgroup, T, of index p in G with center that has index p^3 in G, then A is unique.*
   If p is the smallest prime divisor of $|G|$ and if $|G:A|=p^2$ and $|G:Z(G)|>p^4$ and if G does not possess a subgroup, T, of index p in G with center that has index p^3 in G, then A is unique.

Examples for items 4.a and 4.b in Theorem 2.1 that are chains of length 1 and length 2 (and so in the case of item 4.b, $n=0$ and $m=0$), can be found in [3], see Corollary 2.2, Proposition 2.3, Corollary 2.5, and Proposition 2.6 there.

An example for item 4.b in Theorem 2.1 with $n=1$ and $m=p$ will be given in Section 4.

Another example for item 4.b in Theorem 2.1 is found by taking G to be one of the two extraspecial p-groups of order p^5. $|Z(G)|=p$ (i.e. $|G:Z(G)|=p^4$) and G possesses a maximal self-centralizing elementary abelian subgroup of order p^3 (i.e. of index p^2 in G). The Chermak–Delgado lattice of extraspecial p-groups is known, see Example 2.8 in [6]. And so we have that $\mathcal{CD}(G)$ is isomorphic to the subgroup lattice of an elementary abelian p-group of order p^4. Using the notation of item 4.b in Theorem 2.1, we have that $n=p^3+p^2+p+1$ and $m=(p^2+1)(p^2+p+1)-1$.

Further examples of groups and CD lattices arising from Theorem 2.1 will be explored in the next sections.

3. Dicyclic groups

As a consequence of items 2 and 3 in Theorem 2.1, we classify the Chermak–Delgado lattice of $Dic_{4n}(A)$, where A is an abelian group of order $2n$.

Note that $\mathcal{CD}(Dic_{4n}(A)) = \{A\}$ if and only if $|Dic_{4n}(A) : Z(Dic_{4n}(A))| > 4$, which is equivalent to $|A^2| > 2$. If A is not an elementary abelian 2-group, this means that A is not a direct product of an elementary abelian two-group and $\mathbb{Z}_4$.

Theorem 3.1. *Let A be an abelian group of order $2n$ and $Dic_{4n}(A)$ be the generalized dicyclic group induced by A. Then we have:*

a. *If A is not of type $\mathbb{Z}_2^m \times \mathbb{Z}_4$ with $m \in \mathbb{N}$, then $\mathcal{CD}(Dic_{4n}(A))$ is a chain of length 0, namely $\mathcal{CD}(Dic_{4n}(A)) = \{Dic_{4n}(A)\}$ for $\exp(A) = 2$, and $\mathcal{CD}(Dic_{4n}(A)) = \{A\}$ for $\exp(A) \neq 2$.*
b. *If A is of type $\mathbb{Z}_2^m \times \mathbb{Z}_4$ with $m \in \mathbb{N}$, then $\mathcal{CD}(Dic_{4n}(A))$ is a quasi-antichain of width 3, namely the lattice interval between $Z(Dic_{4n}(A))$ and $Dic_{4n}(A)$.*

As a corollary, we describe the Chermak–Delgado lattice of Dic_{4n}.

Corollary 3.2. *Under the previous notation, we have:*

a. *If $n \neq 2$, then $\mathcal{CD}(Dic_{4n})$ is a chain of length 0, namely $\mathcal{CD}(Dic_{4n}) = \{Dic_{4n}\}$ for $n = 1$, and $\mathcal{CD}(Dic_{4n}) = \{\langle a \rangle\}$ for $n \geq 3$.*
b. *If $n = 2$, then $\mathcal{CD}(Dic_{4n})$ is a quasi-antichain of width 3, namely the lattice interval between $Z(Dic_{4n})$ and Dic_{4n}.*

The following appears in [10].

Theorem 3.3. *Let G be a finite group which can be written as $G = AB$, where A and B are abelian subgroups of relatively prime orders and A is normal. Then*

$$m(G) = |A|^2 |C_B(A)|^2 \text{ and } \mathcal{CD}(G) = \{AC_B(A)\}.$$

Corollary 3.4. *Suppose A is a finite abelian group. There exists a non-abelian finite group G so that $\mathcal{CD}(G) = \{A\}$ if and only if $A \neq 1, A \ncong \mathbb{Z}_2, A \ncong \mathbb{Z}_4$, and $A \ncong \mathbb{Z}_2 \times \mathbb{Z}_4$.*

Proof. $\rightarrow$ Suppose $\mathcal{CD}(G) = \{A\}$ and G is non-abelian. If $A = 1$, then $G = 1$ is abelian, a contradiction. Since A is self-centralizing and normal in G, we have that G/A embeds into $Aut(A)$. If $A \cong \mathbb{Z}_2$, then G/A embeds into $Aut(A)$=1, which contradicts G being non-abelian. If $A \cong \mathbb{Z}_4$, then G/A embeds into $Aut(A)$ which has order 2, and so $|G| = 8$ and $CD(G)$ would not be a chain of length zero by item 2 in Theorem 2.1. Suppose $A \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Then G/A embeds into $Aut(A)$ which is a dihedral group of size 8. And so G is a two-group. Now, $m^*(G) = 64$, and since $G \notin \mathcal{CD}(G)$ and G has a nontrivial center, it follows that $|G| = 16$ and $|Z(G)| = 2$. Note, however, that given $x \in Aut(A)$ of order 2, x must fix at least four elements of A. This is due to the structure of $Aut(A)$ when $A \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. And so $|Z(G)| > 2$, a contradiction.

$\leftarrow$ If $|A|$ is odd, then let $G = A \rtimes \langle x \rangle$ where x inverts each element of A. Then by item 3 in Theorem 2.1, $\mathcal{CD}(G) = \{A\}$. If $|A|$ is even, and A is not of type $\mathbb{Z}_2^m \times \mathbb{Z}_4$ with $m \geq 0$, and $\exp(A) \neq 2$, then $G = Dic_{4n}(A)$ has that $\mathcal{CD}(G) = \{A\}$ by Theorem 3.1. If $A \cong \mathbb{Z}_2^m$ with $m \geq 2$ or $A \cong \mathbb{Z}_2^m \times \mathbb{Z}_4$ with $m \geq 2$, then there exists $x \in Aut(A)$ of order 3. Let $G = A \rtimes \langle x \rangle$. One can apply Theorem 3.3 to get that $\mathcal{CD}(G) = \{A\}$. □

Finally, we present a proposition also applicable to describing $\mathcal{CD}(Dic_{4n})$.

Proposition 3.5. *Let G be a finite group. If $G = HZ(G)$ for some subgroup H of G, then $\mathcal{CD}(G)$ and $\mathcal{CD}(H)$ are lattice isomorphic with $\mathcal{CD}(G) = \{XZ(G) | X \in \mathcal{CD}(H)\}$.*

Proof. Since $G = HZ(G)$, we have that $G/Z(G) \cong H/(Z(G) \cap H) = H/Z(H)$ since $Z(H) \leq Z(G)$. This natural isomorphism between $G/Z(G)$ and $H/Z(H)$ induces a lattice isomorphism between the subgroup intervals $[G:Z(G)]$ and $[H:Z(H)]$, and this lattice isomorphism restricts to a lattice isomorphism between $\mathcal{CD}(G)$ and $\mathcal{CD}(H)$. The result follows. □

4. Metabelian *p*-groups of maximal class

By applying Lemmas 1.1, 1.2, 1.3 and Theorem 2.1, we obtain a classification of the Chermak–Delgado lattices of a metabelian p-groups of maximal class.

Theorem 4.1. *Suppose G is a metabelian p-group of order p^n and of maximal class.*

1. *If $n=3$, then $\mathcal{CD}(G)=\{Z(G),A,A_1,\dots,A_p,G\}$ is a quasi-antichain of width $p+1$, with each A_i abelian.*
2. *If G possesses an abelian subgroup A so that $|G:A|=p$ and $n>3$, then $\mathcal{CD}(G)=\{A\}$.*
3. *Suppose that G does not possess an abelian subgroup A so that $|G:A|=p$. Then we have that $n>4$, and, furthermore:*
   a. *If $|G|=p^5$, then $\mathcal{CD}(G)=\{Z(G),Z(T),G',A_1,\dots,A_p,T,G\}$, where $|T|=p^4, |G'|=|A_i|=p^3$ for each i, $|Z(T)|=p^2$, and $G',A_1,\dots,A_p$ are all abelian and distinct. Also, none of $A_1,\dots,A_p$ are normal in G.*
   b. *If $|G|>p^5$ and G possesses a subgroup T of index p so that $|T:Z(T)|=p^2$, then $\mathcal{CD}(G)=\{Z(T),G',A_1,\dots,A_p,T\}$, where $|T|=p^{n-1}, |G'|=|A_i|=p^{n-2}$ for each i, $|Z(T)|=p^{n-3}$, and $G',A_1,\dots,A_p$ are all abelian and distinct. Also, none of $A_1,\dots,A_p$ are normal in G.*
   c. *If $|G|>p^5$ and G does not possess a subgroup T of index p so that $|T:Z(T)|=p^2$, then $\mathcal{CD}(G)=\{G'\}$.*

Proof. Items 1 and 2 follow directly from items 2 and 3 in Theorem 2.1. The fact that $n>4$ in item 3 follows from Lemma 1.1. Most of Theorem 4.1 part 3 is a straight forward application of Lemma 1.2 and Theorem 2.1 taking $A=G'$. In item 3.a, the fact that there is a unique subgroup, T, of index p in G in $\mathcal{CD}(G)$ is a nontrivial result that follows from Lemma 1.3 which in turn relies on Lemma 4.5.4 in [14]. In items 3.a and 3.b, the fact that none of $A_1,\dots,A_p$ are normal in G follows from Lemma 1.2. □

Using Theorem 4.1, we are able to give some necessary and sufficient conditions under which the Chermak–Delgado lattice of a metabelian p-group of maximal class is a chain of length 0.

Corollary 4.2. *Let G be a metabelian p-group of maximal class. Then $\mathcal{CD}(G)$ is a chain of length 0 if and only if either G possesses an abelian subgroup of index p and $|G:Z(G)|\neq p^2$, or G does not possess abelian subgroups of index p, $|G|>p^5$, and $|M:Z(M)|>p^2$ for any maximal subgroup M of G.*

We provide now a few remarks. The two-groups of maximal class fall into 3 categories: the dihedral groups, the semidihedral groups, and the generalized quaternion groups (see e.g. Theorem 4.1 of [12], II), and each of these possess a cyclic subgroup of index 2. Thus, the CD lattices of these groups fall under the umbrella of items 2 and 3 in Theorem 2.1.

Xue, Lv, and Chen provide sufficient conditions for when p-groups of maximal class have CD lattices that fall under the umbrella of item 3 in Theorem 2.1 (see Theorem 2.9 of [17]):

Proposition 4.3. *Let G be a finite p-group of order p^n and of maximal class, where $p\geq 5$ and $n>2p$. If $C_G(H)=Z(H)$ for every non-abelian subgroup H of G, then G possesses an abelian subgroup of index p.*

Note that the finite non-abelian groups satisfying the condition in Proposition 4.3 are called *CGZ-groups*. Xue, Lv, and Chen show that if G is a p-group of maximal class and G is a CGZ-group, then G is metabelian (see Proposition 2.6 of [17]).

We end with examples of metabelian p-groups of maximal class for each of the three cases in item 3 of Theorem 4.1. For the case a) such an example is presented in [17]:

$$G = \langle a, b, c, d | a^{p^2} = b^p = c^p = d^p = 1, [b, a] = c, [c, a] = d, [c, b] = [d, b] = a^p, [d, a] = 1 \rangle,$$

where $p \geq 5$.

For case (b), we present $SmallGroup(3^6, 99)$ from GAP:

$$G = \langle a, b, c, d, e, f | a^3 = e^3 = f^3 = 1, b^3 = c^2 d, c^3 = e^2 f, d^3 = f^2, [b, a] = c, [c, a] = d, [d, a] = e,$$
$$[e, a] = [c, b] = f, [f, a] = [d, b] = [e, b] = [f, b] = [d, c] = [e, c] = [f, c] = [e, d] = [f, d] = [f, e] = 1 \rangle.$$

For case (c), we present $SmallGroup(5^6, 651)$ from GAP:

$$G = \langle a, b, c, d, e, f | a^5 = c^5 = d^5 = e^5 = f^5 = 1, b^5 = f^3, [b, a] = c, [c, b] = d, [c, a] = [d, b] = e,$$
$$[d, a] = [e, b] = f, [e, a] = [f, a] = [f, b] = [d, c] = [e, c] = [f, c] = [e, d] = [f, d] = [f, e] = 1 \rangle.$$

Acknowledgments

The authors are grateful to the reviewer for its remarks which improve the previous version of the article.

Disclosure statement

No potential conflict of interest was reported by the authors.

ORCID

Ryan McCulloch http://orcid.org/0000-0002-3059-5039

References

[1] An, L., Brennan, J. P., Qu, H., Wilcox, E. (2015). Chermak-Delgado lattice extension theorems. *Comm. Algebra* 43(5):2201–2213. DOI: 10.1080/00927872.2014.889147.

[2] Brewster, B., Wilcox, E. (2012). Some groups with computable Chermak-Delgado lattices. *Bull. Aust. Math. Soc.* 86(1):29–40. DOI: 10.1017/S0004972712000196.

[3] Brewster, B., Hauck, P., Wilcox, E. (2014). Groups whose Chermak-Delgado lattice is a chain. *J. Group Theor.* 17:253–279.

[4] Brewster, B., Hauck, P., Wilcox, E. (2014). Quasi-antichain Chermak-Delgado lattices of finite groups. Arch. Math. 103(4):301–311. DOI: 10.1007/s00013-014-0696-3.

[5] Chermak, A., Delgado, A. (1989). A measuring argument for finite groups. *Proc. Amer. Math. Soc.* 107(4): 907–914. DOI: 10.2307/2047648.

[6] Glauberman, G. (2006). Centrally large subgroups of finite p-groups. *J. Algebra* 300(2):480–508. DOI: 10.1016/j.jalgebra.2005.11.033.

[7] Isaacs, I. M. (2008). *Finite Group Theory*. Providence, RI: American Mathematical Society.

[8] McCulloch, R. (2017). Chermak-Delgado simple groups. *Comm. Algebra* 45(3):983–991. DOI: 10.1080/00927872.2016.1172623.

[9] McCulloch, R. (2018). Finite groups with a trivial Chermak-Delgado subgroup. *J. Group Theor.* 21(3): 449–461. DOI: 10.1515/jgth-2017-0042.

[10] McCulloch, R., Tărnăuceanu, M. (2018). Two classes of finite groups whose Chermak-Delgado lattice is a chain of length zero. *Comm. Algebra* 46(7):3092–3096. DOI: 10.1080/00927872.2017.1404090.

[11] Schmidt, R. (1994). *Subgroup Lattices of Groups, de Gruyter Expositions in Mathematics 14*. Berlin: de Gruyter.

[12] Suzuki, M. (1982). *Group Theory, I, II*. Berlin: Springer Verlag.

[13] Tărnăuceanu, M. (2017). The Chermak-Delgado lattice of ZM-groups. *Results Math.* 72(4):1849–1855. DOI: 10.1007/s00025-017-0735-z.

[14] Vieira, L. S. (2017). *On p-adic fields and p-groups*, Ph.D. Thesis, University of Kentucky.

[15] Wilcox, E. (2016). Exploring the Chermak-Delgado lattice. *Math. Magazine* 89(1):38–44. DOI: 10.4169/math.mag.89.1.38.

[16] Xu, M., An, L., Zhang, Q. (2008). Finite p-groups all of whose non-abelian proper subgroups are generated by two elements. *J. Algebra* 319(9):3603–3620. DOI: 10.1016/j.jalgebra.2008.01.045.

[17] Xue, H., Lv, H., Chen, G. (2014). On a special class of finite p-groups of maximal class. *Italian J. Pure Appl. Math.* 33:279–284.

[18] Morresi Zuccari, A., Russo, V., Scoppola, C. M. (2018). The Chermak-Delgado measure in finite p-groups. *J. Algebra* 502:262–276. DOI: 10.1016/j.jalgebra.2018.01.030.